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ON THE MOMENTUM EQUATIONS IN DISPERSED TWO-PHASE SYSTEMS

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Abstract—For so-called dispersed two-phase flow systems the equations of motion are derived for each phase separately. The equations are developed using a minimum of mathematics by applying mass and momentum balances respectively over a small cubic volume element. Special attention is paid to those particles which are cut by the boundaries of this control volume element. In fact these particles are treated by means of two methods, both methods giving the same final result.

1. INTRODUCTION

Two-phase flow systems can roughly be divided in: (a) segregated two-phase flow systems, in which wall effects play a dominating role and in which one phase is mainly segregated and as such flowing along the walls; (b) dispersed two-phase flow systems, in which wall effects are only of minor importance and in which one phase—the dispersed phase—is present as small particles, drops or bubbles anywhere in the other phase—the continuous phase.

This paper will deal only with dispersed two-phase flow systems of which the scale of dispersion is at least two orders of magnitude ($10^2\times$) smaller than the scale of the apparatus, equipment or natural surroundings in which it is present or through which it moves.

A derivation of the momentum equations for each phase separately will be given for the case that no thermodynamic or chemical processes occur. Further restrictions are that neither inversion of the dispersion occurs nor coalescing or continued dispergating of the dispersed phase while the phase ratio should be a continuous function in space and time.

If coalescing and redispersion are excluded so that each particle maintains its identity then a specific velocity can be attributed to each particle which is the velocity of its mass centre. Of course, inside the particle velocity differences may occur (e.g. the particle might rotate around its mass centre if it is a solid or an internal vortex ring might occur if the particle is fluid). It is believed, however, that these velocity differences only affect the constitutive equations, notably that equation which relates the so-called slip force to the slip velocity. Hence, in this paper the velocity differences inside the dispersed particles will be ignored (see also discussion).

Earlier derivations of the momentum equations have been given by, e.g. Van Deemter & Van der Laan (1961), Hinze (1962), Jackson (1963), Anderson & Jackson (1967), Drew & Segel (1971) and Ishii (1975). There is, however, still disagreement about the correct shape of these equations.

2. PRESSURE AND SHEAR STRESS INSIDE THE DISPERSED PARTICLES

In a continuous phase shear stress and fluid pressure will be continuous in space and time. When we pass the interface between the two phases, however, this might not be true. When both phases are fluid the interfacial tension γ can generally not be ignored and it is this tension which causes each separate particle to keep its identity. It also causes pressure p_d inside the particle to differ from the continuous phase pressure p. For a spherical particle it holds that $p - p_d = 2\gamma/a$ in which a is the radius of the particle. It is obvious that in a cross-section of the system normal to the direction of flow the pressure p_d can have different values in the individual particles, depending on their sizes. Hence, it is concluded that p_d has no relation at

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all to the flow of the dispersed system and therefore is irrelevant for the description of the flow equations.

In a similar way it can be reasoned that the shear stress $\underline{\tau}_d$ inside the particle is not related directly to the motion of the dispersed particles. Hence $\underline{\tau}_d$ is not to be confused with the stress tensor $\underline{\sigma}_d$ which arises from interaction and exchange of momentum between separate individual particles either by direct permanent contact between these particles or by collisions, attraction or repulsion if there is no permanent contact or no contact at all.†

3. AVERAGING OF VARIABLES

In a dispersed two-phase system momentary point variables like velocity, pressure, etc. vary due to macroscopic variations which determine the overall macroscopic behaviour of the dispersion at the scale of the macroscopic equipment which contains the dispersion.

On top of these macroscopic variations the variables also fluctuate in time and in space due to microscopic variations at the scale of the dispersed particles themselves. These variations arise from the complicated paths of individual particles and from the distortion of fluid streamlines around and between the particles (microturbulence).

In deriving manageable mass and momentum balances, therefore, an averaging procedure should be introduced in such a way that any momentary point property a' can be conceived as the superposition of its local mean value a and a fluctuation a'' which varies in time and in space. This means that the average of a'' should be zero: (1) when—at a certain moment—averaged over the control volume V_{r_0} or over the control surface element A_{r_0}; (2) when—at a certain location—averaged over a sufficiently long time Δt.

The size r_0 of this control volume or surface element should be large compared to the scale l_d of the dispersion but small compared to the size L of the macroscopic equipment ($l_d \ll r_0 \ll L$). In the same way the integration time Δt should be long compared to the passage time t_d of a single particle ($t_d = l_d/v_d$) and short compared to the relaxation time T of the macroscopic variations ($t_d \ll \Delta t \ll T$).

That indeed a'' when averaged over the control surface A_{r_0} becomes zero and becomes zero as well when averaged over Δt follows from the consideration that in a stationary flow field the spectrum distribution of the fluctuations over the surface A_{r_0} must be the same as the spectrum distribution over the time Δt.

The above means that when, e.g. the product $\rho_d v''_{dx} v''_{dy}$ is averaged over A_{r_0} we obtain exactly the same value as when averaged over Δt. It will be denoted as $\rho_d v''_{dx} v''_{dy}$.

Hence the so-called Reynolds stresses due to microscopic turbulence can be obtained from surface averaging as well as from time averaging.

Hence also when surface averaging has been carried out time averaging of the surface average variables becomes meaningless.

Of course all this does not hold for macroscopic turbulence which in this paper is left out of consideration.

3.1 *The averaging procedure of Anderson & Jackson*

Anderson & Jackson (1967) applied an averaging procedure in which local mean variables at point $\underline{r}$ are derived by means of integration of these variables over a weighting function $g(l)$ where $l = |\underline{r} - \underline{r}_1|$.

This weighting function should further satisfy the requirements that it is: (a) analytical and >0 for all l; (b) monotonically decreasing with increasing l; (c) normalised: $4\pi \int_0^\infty g(l) l^2 \, dl = 1$; and finally (d) that it has a radius r_0 defined by $4\pi \int_0^{r_0} g(l) l^2 \, dl = \frac{1}{2}$ where $l_d \ll r_0 \ll L$.

This indeed seems a very elegant method especially since it avoids the problem how to deal

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†$\underline{\sigma}_d$ also differs in character from $\underline{\tau}_d$ since it contains the isotropic interparticle pressure (equivalent to the continuous phase pressure p). This isotropic interparticle pressure will not be nominated here.

with particles which are cut by the boundaries of a reference volume over which the mass and momentum balances are taken. On the other hand, in the elaboration of the above integrals assumptions have to be made on the variation of the variables which are not always justified or even seem to be contradictory (as explained in appendix I).

3.2 *Integration over a cubic volume element*

We propose to apply the well-known method of balancing over a cubic volume element where the separate contributions to these balances now have to be calculated by integration over the volume and the faces of this volume element respectively.

Again the size of this volume element should be large compared to the scale of the dispersion and small compared to the size of the apparatus.

In this method therefore fluxes are averaged (with a weighting function $g = 1$) over the boundary planes of the volume element, e.g.

$$v_x(x_1, y, z) = \frac{1}{\Delta y \Delta z} \int_{\Delta y, \Delta z} v_x'(x_1, y, z)\, \mathrm{d}y\, \mathrm{d}z \qquad [1]$$

while concentrations are averaged (again with a weighting function $g = 1$) over the volume of the element, e.g. for the momentum concentration:

$$(\rho v_x) = \frac{1}{\Delta x \Delta y \Delta z} \int_{\Delta x, \Delta y, \Delta z} (\rho' v_x')\, \mathrm{d}x \mathrm{d}y\, \mathrm{d}z. \qquad [2]$$

Indeed the requirement that the size of the volume element should be large compared to the scale of the dispersion forbids us to take the limit as Δx, Δy and Δz approach to zero. If, however, the size of the volume element is indeed small compared to the size of the apparatus it is reasonable to assume that within the volume element all variables change linearly so that, e.g.

$$v_x(x_1 + \Delta x) = v_x(x_1) + \left(\frac{\partial v_x}{\partial x}\right)_{x_1} \Delta x.$$

With this assumption the balances can be easily elaborated.

3.3 *Application to dispersed two-phase systems*

For a dispersed two-phase system the above indicated method must be elaborated taking into account that each phase is not present everywhere. First of all a mathematical formulation must be given of the volume fraction of continuous phase, generally named the porosity ϵ. The volume fraction of dispersed phase then is $(1 - \epsilon)$.

This porosity of course can only be defined at a scale large compared to the scale of dispersion since *at* the scale of dispersion the volume fraction of each phase is either one or zero. This also means that it does not make sense to distinguish between point values of the porosity (ϵ') and local mean values (ϵ) as is proposed for all other variables.

A satisfactory definition of the porosity runs:

$$\epsilon = \frac{1}{\Delta x \Delta y \Delta z} \int\limits_{\substack{\Delta x, \Delta y, \Delta z \\ \text{cont.phase}}} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \qquad [3]$$

where the integration is extended over all points which are occupied by the continuous phase.

In averaging fluxes over phases this definition is not adequate. To this end we now suppose that the control volume element is split up into a large number of thin slabs parallel to the plane

$x = x_1$ where the thickness of these slabs is small compared to the size of the dispersed particles.

The intersection of these slabs by the dispersed particles then can be conceived as being constant (independent of x, see figure 1) over this thickness. It follows that ϵ does not change if we put:

$$\epsilon = \frac{1}{\Delta x}\int_{\Delta x}\left\{\frac{1}{\Delta y\Delta z}\int\limits_{\substack{\Delta y,\Delta z\\ \text{cont.phase}}} \mathrm{d}y\,\mathrm{d}z\right\}\mathrm{d}x = \frac{1}{\Delta y\Delta z}\int\limits_{\substack{\Delta y,\Delta z\\ \text{cont.phase}}} \mathrm{d}y\,\mathrm{d}z \qquad [4]$$

Of course similar expressions can be derived for slabs in the remaining directions. These expressions will be used in elaborating fluxes.

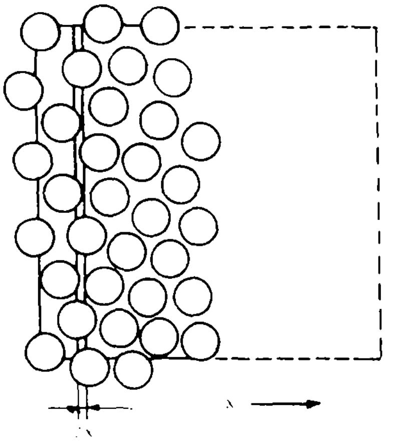

Figure 1.

In accordance with the foregoing definition a flux a_c' of the continuous phase through the plane $\Delta y\Delta z$ at $x = x_1$ now will be averaged as follows:

$$\epsilon a_c(x_1, y_1, z_1) = \frac{1}{\Delta y\Delta z}\int\limits_{\substack{\Delta y,\Delta z\\ \text{cont.phase}}} a_c'(x_1, y, z)\,\mathrm{d}y\,\mathrm{d}z \qquad [5]$$

and similarly a flux a_d' of the dispersed phase as

$$(1-\epsilon)a_d(x_1, y_1, z_1) = \frac{1}{\Delta y\Delta z}\int\limits_{\substack{\Delta y,\Delta z\\ \text{disp.phase}}} a'(x_1, y, z)\,\mathrm{d}y\,\mathrm{d}z. \qquad [6]$$

If $\epsilon = 1$ (no disp. phase present) or $\epsilon = 0$ (no cont. phase present) these definitions give the desired result. In case a_c' (resp. a_d') is constant over the control volume, integration over Δx immediately gives the original definition of the porosity ϵ.

A concentration in the continuous phase or a body force I acting upon this phase will be averaged according to:

$$\epsilon I = \frac{1}{\Delta x\Delta y\Delta z}\int\limits_{\substack{\Delta x,\Delta y,\Delta z\\ \text{cont.phase}}} I'\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad [7]$$

which again agrees with the original definition of the porosity.

Finally we must decide what to do with those particles which are cut by the boundaries of the control volume element. These particles will belong to this volume element only partly and, therefore, summing up all momentum flows and all forces acting upon this element we are dealing with p_d and τ_d which, as mentioned before, have no direct relation with the motion of the dispersed phase. This difficulty seems too serious to be ignored, as was done by Van Deemter & Van der Laan (1961).

Two methods seem to be appropriate to handle this difficulty. Since both methods should finally result into the same equations we thus obtain a check on these equations.

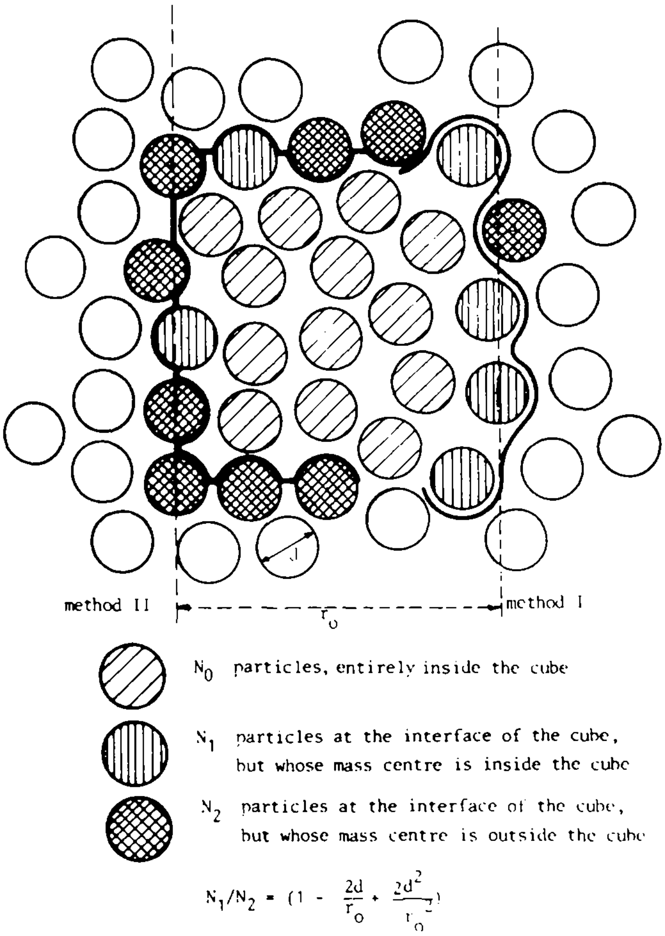

Figure 2.

When we consider all particles which are cut by the faces of the control volume element we can distinguish two groups: a group N_1 whose mass centre is lying within the control volume element or in its faces, and a group N_2 whose mass centre is lying outside the control volume element (see figure 2). On an average N_2 is somewhat larger than N_1, their ratio N_1/N_2 approaching $(1 - 2d/r_0)$ in which d is the average diameter of the dispersed particles and r_0 the length of the edges of the control volume element. Ultimately it will appear that this difference between N_2 and N_1 has no effect on our derivations.

Now in *method I* the boundary planes of the volume element are conceived to be flexible and to run through the continuous phase around the particles N_1 and N_2 in such a way that the particles N_1 are entirely inside the control volume element and the particles N_2 entirely outside this volume element. In this way the average location of the boundary planes remains the same while also the volume of both phases inside the volume element is not affected. External forces (except body forces) are all acting on the continuous phase. It will be clear, however, that this method can be applied only if the volume concentration of the dispersed phase is low and that it certainly can not be applied if there is permanent contact between the dispersed particles.

In *method II* it is supposed that the control volume element again contains the particles N_0 and N_1 but not the particles N_2. The boundary planes are following the exact mathematical planes (x = constant, y = constant or z = constant) as long as they run through the continuous phase (see figure 2, the fat line), but follow the 'external' surface (S_e) of the particles N_1 and the 'internal' surface (S_i) of the particles N_2 (see section 4.3). This method implies that forces working on particles which are intersected by the flat boundary planes now partly belong to external forces and partly to internal forces.

How both methods will proceed will be seen in the next three sections.

We will confine ourselves to the case that the densities of both participating phases may be considered constant while phase transitions may be ignored.

We choose the following notation:

Continuous phase		Dispersed phase
ϵ	volume fraction	$1-\epsilon$
ρ_c	density	ρ_d
$\underline{v}_c$	local mean velocity	$\underline{v}_d$
$\underline{n}_c = \epsilon\rho_c\underline{v}_c$	mass flux	$\underline{n}_d = (1-\epsilon)\rho_d\underline{v}_d$

4. THE MOMENTUM BALANCES

Most generally the law of conservation of momentum for each phase runs:

$$\left[\begin{array}{c}\text{rate of accumulation}\\ \text{of momentum of}\\ \text{phase considered}\end{array}\right] = \left[\begin{array}{c}\text{rate in of momentum}\\ \text{for that phase by}\\ \text{convection}\end{array}\right] - \left[\begin{array}{c}\text{rate out of momentum}\\ \text{for that phase by}\\ \text{convection}\end{array}\right]$$

$$+ \left[\begin{array}{c}\text{sum of external}\\ \text{forces acting upon}\\ \text{that phase}\end{array}\right] + \left[\begin{array}{c}\text{sum of internal}\\ \text{forces acting upon}\\ \text{that phase}\end{array}\right].$$

It must be recognized that the internal forces also include the interaction force $\underline{F}_{dc}$ which the dispersed phase exerts on the continuous phase (per unit volume of dispersed system) or similarly the force $-\underline{F}_{dc}$ which is exerted upon the dispersed phase.

4.1 *The momentum balance of the continuous phase* (*method I*)

The evaluation of the net rate of momentum by *convection* becomes:

$$\left[\begin{array}{c}\text{rate in by convection}\\ \text{of momentum in}\\ x\text{-direction}\end{array}\right] = \int\limits_{\substack{\Delta y,\Delta z\\ \text{cont.phase}\\ \text{at } x=x_1}} \rho_c v'_{cx} v'_{cx}\,\mathrm{d}y\,\mathrm{d}z + \oint\limits_{\substack{\Delta x,\Delta z\\ \text{cont.phase}\\ \text{at } y=y_1}} \rho_c v'_{cx} v y\,\mathrm{d}x\,\mathrm{d}z$$

$$+ \int\limits_{\substack{\Delta x,\Delta y\\ \text{cont.phase}\\ \text{at } z=z_1}} \rho_c v'_{cx} v'_{cz}\,\mathrm{d}x\,\mathrm{d}y \qquad [8]$$

$$\left[\begin{array}{c}\text{rate out by convection}\\ \text{of momentum in}\\ x\text{-direction}\end{array}\right] = \int\limits_{\substack{\Delta y,\Delta z\\ \text{cont.phase}\\ \text{at } x=x_1+\Delta x}} \rho_c v'_{cx} v'_{cx}\,\mathrm{d}y\,\mathrm{d}z + \int\limits_{\substack{\Delta x,\Delta z\\ \text{cont.phase}\\ \text{at } y=y_1+\Delta y}} \rho_c v'_{cx} v'_{cy}\,\mathrm{d}x\,\mathrm{d}z$$

$$+ \int\limits_{\substack{\Delta x,\Delta y\\ \text{cont.phase}\\ \text{at } z=z_1+\Delta z}} \rho_c v'_{cx} v'_{cz}\,\mathrm{d}x\,\mathrm{d}y. \qquad [9]$$

Whether we apply method I or II for the faces of the volume element has as yet not made any difference with respect to the convective transport through these faces. Further elaborating the two above contributions gives us the net convection contribution. Also here the products of two fluctuations generally do not average out but contribute to the total momentum transport in a way comparable to the Reynolds stresses in turbulent transport. Hence a term $\underline{\nabla \cdot \underline{\underline{R}}_c}$ enters the momentum equation of the continuous phase where we have introduced $\underline{\underline{R}}_c = \overline{\epsilon\rho_c\underline{v}''_c\underline{v}''_c}$.

Now the net convection contributions in the x-direction runs as:

$$-\rho_c\left\{\frac{\partial}{\partial x}(\epsilon v_{cx}v_{cx})+\frac{\partial}{\partial y}(\epsilon v_{cx}v_{cy})+\frac{\partial}{\partial z}(\epsilon v_{cx}v_{cz})\right\}\Delta x\Delta y\Delta z$$

$$-\rho_c\left\{\frac{\partial}{\partial x}(\overline{\epsilon v''_{cx}v''_{cx}})+\frac{\partial}{\partial y}(\overline{\epsilon v''_{cx}v''_{cy}})+\frac{\partial}{\partial z}(\epsilon v''_{cx}v''_{cz})\right\}\Delta x\Delta y\Delta z$$

$$=-\{\rho_c[\nabla\cdot(\epsilon\underline{v}_c\underline{v}_c)]_x+[\nabla\cdot\underline{\underline{R}}_c]_x\}\Delta x\Delta y\Delta z. \qquad [10]$$

The first external forces to be evaluated are those exerted by the *shear stress* $\underline{\underline{\tau}}_c$, Now it is important that we deal with method I since the faces through which the shear occurs now are entirely in the continuous phase and hence in averaging over these faces the porosity does not come in:

$$\left[\begin{array}{c}\text{rate in by shear}\\ \text{of momentum in}\\ x\text{-direction}\end{array}\right]=\int\limits_{\substack{\Delta y,\Delta z,\\ \text{at } x_1}}\tau'_{cxx}\,\mathrm{d}y\,\mathrm{d}z+\int\limits_{\substack{\Delta x,\Delta z,\\ \text{at } y_1}}\tau'_{cxy}\,\mathrm{d}x\,\mathrm{d}z+\int\limits_{\substack{\Delta x,\Delta y,\\ \text{at } z_1}}\tau'_{cxz}\,\mathrm{d}x\,\mathrm{d}y \qquad [11]$$

$$\left[\begin{array}{c}\text{rate out by shear}\\ \text{of momentum in}\\ x\text{-direction}\end{array}\right]=\int\limits_{\substack{\Delta y,\Delta z,\\ \text{at } x_1+\Delta x}}\tau'_{cxx}\,\mathrm{d}y\,\mathrm{d}z+\int\limits_{\substack{\Delta x,\Delta z,\\ \text{at } y_1+\Delta y}}\tau'_{cxy}\,\mathrm{d}x\,\mathrm{d}z+\int\limits_{\substack{\Delta x,\Delta y,\\ \text{at } z_1+\Delta z}}\tau'_{cxz}\,\mathrm{d}x\,\mathrm{d}y \qquad [12]$$

so that the net external force by shear in the x-direction becomes:

$$-\left\{\frac{\partial}{\partial x}(\tau_{cxx})+\frac{\partial}{\partial y}(\tau_{cxy})+\frac{\partial}{\partial z}(\tau_{cxz})\right\}\Delta x\Delta y\Delta z=-[\nabla\cdot\underline{\underline{\tau}}_c]_x\Delta x\Delta y\Delta z. \qquad [13]$$

The other external forces acting upon the continuous phase contained in the control volume element include the action of the pressure p and the body force due to gravity.

The external forces exerted in the x-direction by the *pressure* are acting only upon the faces Δy, Δz. Applying method I these forces entirely act upon the continuous phase and again the porosity does not come in. Their combined contribution is

$$-[\nabla p]_x\Delta x\Delta y\Delta z.$$

The *body force* is found directly by integration of the gravity acceleration over all the mass of the continuous phase contained in the control volume element, and hence

$$=\int\limits_{\substack{\Delta x,\Delta y,\Delta z\\ \text{cont.phase}}}\rho_c\underline{g}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=\epsilon\rho_c\underline{g}\Delta\Delta y\Delta z$$

The component of this force in the x-direction runs as $\epsilon\rho_c g_x\Delta x\Delta y\Delta z$.

Finally we must evaluate the *internal force* exerted by the dispersed phase upon the continuous phase, the so-called interaction force $\underline{F}_{dc}$. It is found by summation of all forces which the individual particles contained in the control volume exert upon the continuous phase. Hence

$$\underline{F}_{dc}=\frac{1}{r_0^3}\sum_{\text{all particles}}\underline{F}'_{pc}$$

in which:

$$\underline{F}'_{pc}=\int\limits_{S_p}\underline{n}\cdot(p'\underline{\underline{\delta}}+\underline{\underline{\tau}}'_c)\,\mathrm{d}S=\int\limits_{V_p}(\nabla p'+\nabla\cdot\underline{\underline{\tau}}'_c)\,\mathrm{d}V_p. \qquad [14]\dagger$$

In the latter integral $\underline{\underline{\tau}}'_c$ inside the particle is the imaginary continuation of $\underline{\underline{\tau}}'_c$ outside the particle.

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†Since $\underline{F}'_{pc}$ is the interaction force of one single particle it is a point variable and therefore is given a single prime. After summation over all particles in the control volume this prime disappears in the mean interaction force $\underline{F}_{dc}$.

The total interaction force of all particles contained in the control volume element upon the continuous phase, therefore, is

$$\Delta x\Delta y\Delta z\underline{F}_{dc} = \sum_{\text{all particles}} \underline{F}'_{pc} = \sum_{\text{all particles}} \int_{V_p} (\nabla p' + \nabla\cdot\underline{\underline{\tau}}'_c)\,\mathrm{d}V_p$$

$$= \int_{\substack{\Delta x,\Delta y,\Delta z,\\ \text{disp.phase}}} (\nabla p' + \nabla\cdot\underline{\underline{\tau}}'_c)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \Delta x\Delta y\Delta z\{(1-\epsilon)(\nabla p + \nabla\cdot\underline{\underline{\tau}}_c) + \underline{F}_1\} \qquad [15]$$

where we have introduced:

$$\underline{F}_1 = \frac{1}{\Delta x\Delta y\Delta z}\int_{\substack{\Delta x,\Delta y,\Delta z,\\ \text{disp.phase}}} (\nabla p'' + \nabla\cdot\underline{\underline{\tau}}''_c)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \qquad [16]$$

This force $\underline{F}_1$ can not be further elaborated (see Discussion). In the x-direction we have of course

$$F_{dc,x} = (1-\epsilon)[\nabla p + \nabla\cdot\underline{\underline{\tau}}_c]_x + F_{1x}. \qquad [17]$$

The *rate of accumulation* is easily found:

$$\begin{bmatrix}\text{rate or accumulation}\\ \text{of momentum in}\\ x\text{-direction}\end{bmatrix} = \frac{\partial}{\partial t}\int_{\substack{\Delta x,\Delta y,\Delta z\\ \text{cont.phase}}} \rho_c v'_{cx}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \rho_c\frac{\partial}{\partial t}(\epsilon v_{cx})\Delta x\Delta y\Delta z. \qquad [18]$$

Now equating according to the law of conservation of momentum we find after dividing by $\Delta x\Delta y\Delta z$:

$$\rho_c\frac{\partial}{\partial t}(\epsilon v_{cx}) = -\rho_c[\nabla\cdot(\epsilon\underline{v}_c\underline{v}_c)]_x - [\nabla\cdot\underline{\underline{R}}_c]_x - [\nabla\cdot\underline{\underline{\tau}}_c]_x - [\nabla p]_x + \epsilon\rho_c g_x + (1-\epsilon)[\nabla p + \nabla\cdot\underline{\underline{\tau}}_c]_x + F_{1x}$$

$$= -\rho_c[\nabla\cdot(\epsilon\underline{v}_c\underline{v}_c)]_x - [\nabla\cdot\underline{\underline{R}}_c]_x - \epsilon[\nabla\cdot\underline{\underline{\tau}}_c]_x - \epsilon[\nabla p]_x + \epsilon\rho_c g_x + F_{1x}. \qquad [19]$$

Similar expressions are found for momentum in the y-direction and that in the z-direction. Therefore we can formulate the general vector equation:

$$\rho_c\left[\frac{\partial}{\partial t}(\epsilon\underline{v}_c) + \nabla\cdot(\epsilon\underline{v}_c\underline{v}_c)\right] = -\epsilon\nabla\cdot\underline{\underline{\tau}}_c - \nabla\cdot\underline{\underline{R}}_c - \epsilon\nabla p + \epsilon\rho_c\underline{g} + \underline{F}_1. \qquad [20]$$

The l.h.s. can be further worked out by means of the continuity equation which finally gives:

$$\epsilon\rho_c\frac{D}{Dt}(\underline{v}_c) = \epsilon\rho_c\left(\frac{\partial\underline{v}_c}{\partial t} + \underline{v}_c\cdot\nabla\underline{v}_c\right) = -\epsilon\nabla\cdot\underline{\underline{\tau}}_c - \nabla\cdot\underline{\underline{R}}_c - \epsilon\nabla p + \epsilon\rho_c\underline{g} + \underline{F}_1. \qquad [21]$$

4.2 *The momentum balance of the dispersed phase* (*method I*)

We can proceed in a similar way to form the momentum balance equation of the dispersed phase. Following method I it will be clear that—since all particles are entirely surrounded by continuous phase within the control volume element—the only external force which now comes in is the body force exerted by *gravitation.*

Of course the fact that there are no particles which are intersected by the boundary planes can not mean that there is no *convective* contribution to momentum transport. This seems a contradiction which can be solved by means of the flexible boundary planes which we suppose to jump around a dispersed particle moving in the direction of the control volume when it is close enough, or just reverse to retreat from a particle moving away from the control volume when it is far enough off.

Again a contribution $\nabla\cdot\underline{\underline{R}}_d$ to the momentum transport is found as a consequence of the *fluctuations* of the dispersed particle velocity $\underline{v}_d$:

$$\underline{\underline{R}}_d = \rho_d\{\overline{(1-\epsilon)\underline{v}''_d\underline{v}''_d}\}.$$

Now, however, the dispersed nature of the individual particles manifests itself. In applying method I we supposed that there is no permanent contact between the dispersed particles (the porosity ϵ is close to one) and only occasionally collisions may happen. In that case there can not be an effective transport of momentum from one particle to another and also the fluctuations of the dispersed particle velocity can not arise from the dispersed phase itself but must be induced by the fluctuations in the continuous phase velocity. The gain or loss of momentum of a dispersed particle which is related to the fluctuation of its velocity must come from and finally made up by the surrounding continuous phase. This means that for each particle separately the product ($\underline{v}''_d\underline{v}''_d$) when averaged over time must disappear.† Since there also is no transport of one particle to another $\underline{\underline{R}}_d$ must be $= 0$. Of course this no longer holds true at lower porosities when there is permanent contact or occasional collisions can no longer be neglected. Hence, $\underline{\underline{R}}_d \neq 0$ at lower porosity.

In both cases ($\underline{\underline{R}}_d = 0$ at high porosity and $\neq 0$ at lower porosity) one might argue that through the interaction force $\underline{F}_{dc}$ and especially through the part $\underline{F}_s$ of that force which is directly related to the slip velocity $\underline{v}_s$ between the two phases ($\underline{v}_s = \underline{v}_d - \underline{v}_c$) momentum of the dispersed phase is transferred to the continuous phase and hence one would wonder whether the product $\rho_d\underline{v}''_d\underline{v}''_d$ or part of it would not contribute to the momentum transport in the continuous phase.

As mentioned however, this should happen by the action of the interaction force $\underline{F}_s$ and hence is entirely accounted for by the fluctuation $\underline{F}''_s$ of this interaction force. Now at low Reynolds numbers (related to the slip velocity $\underline{v}_s$) the point value $\underline{F}'_s = \underline{F}_s + \underline{F}''_s$ can be expected to depend linearly on the point slip velocity $\underline{v}'_s = \underline{v}_s + \underline{v}''_s$ (where $\underline{v}''_s = \underline{v}''_d - \underline{c}''$). Hence on averaging $\underline{F}''_s$ will disappear and hence also the contribution of the product $\rho_d\underline{v}''_d\underline{v}''_d$ to the momentum transport in the continuous phase. At higher Reynolds numbers this of course is no longer true and in that case we must expect a rather strong increase of $\underline{F}_s$ which probably can be brought back to an increase of the specific force $\underline{F}_1$.

Now returning to the momentum balance of the dispersed phase there again is only one *internal force*, viz. $-\underline{F}_{dc}$, which is just the opposite of the interaction force $\underline{F}_{dc}$ calculated in 4.1.

Therefore, in the case the porosity is close to unity (as mentioned before, only in this case the application of method I is allowed), the momentum balance of the dispersed phase becomes:

$$(1-\epsilon)\rho_d \frac{D}{Dt}(\underline{v}_d) = (1-\epsilon)\rho_d \left\{\frac{\partial \underline{v}_d}{\partial t} + \underline{v}_d \cdot \nabla \underline{v}_d\right\} = (1-\epsilon(\rho_d g - \underline{F}_{dc}$$

$$= (1-\epsilon)\rho_d \underline{g} - (1-\epsilon)[\nabla p + \nabla \cdot \underline{\underline{\tau}}_c] - \underline{F}_1. \qquad [22]$$

4.3 *The momentum balances according to method II*

Following method II there is no restriction anymore concerning the volume fraction of dispersed phase and so there is no need to assume that the dispersed particles do not touch each other. The faces of the control volume element are now cut in two parts: one part of fraction ϵ which is occupied by continuous phase and a part of fraction $1-\epsilon$ occupied by dispersed phase. This means that the external forces acting on the phases have changed, except of course the body forces: (1) the rates in and out caused by the shear stresses and pressure on the continuous phase are a factor ϵ smaller; (2) the remainder of the external forces is now acting on the dispersed phase.

Also the interaction force of the dispersed phase on the continuous phase, now called $\underline{F}^*_{dc}$,

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†As mentioned before (section 3) averaging over time gives the same result as averaging over the control surface area A_{r_0}. A_{r_0}.

has changed as will be seen in the following. Of course the fluxes by convection as well as the accumulation rate and the body forces are still the same as derived in applying method I.

Comparing with method I we can make the following balances (see figure 2).

$$\left[\begin{array}{c}\text{external stresses}\\ \text{on cont. phase}\end{array}\right]_{\mathrm{I}} = \left[\begin{array}{c}\text{external stresses}\\ \text{on cont. phase}\end{array}\right]_{\mathrm{II}} + \sum_{N_2} \int_{S_i} \underline{n} \cdot (p'\underline{\underline{\delta}} + \underline{\underline{\tau}}_c')\,\mathrm{d}S - \sum_{N_1} \int_{S_e} \underline{n} \cdot (p'\underline{\underline{\delta}} + \underline{\underline{\tau}}_c')\,\mathrm{d}S \tag{23}$$

$$\left[\begin{array}{c}\text{external stresses}\\ \text{on disp. phase}\end{array}\right]_{\mathrm{I}} = \left[\begin{array}{c}\text{external stresses}\\ \text{on disp. phase}\end{array}\right]_{\mathrm{II}} + \sum_{N_1} \int_{S_e} \underline{n} \cdot (p'\underline{\underline{\delta}} + \underline{\underline{\tau}}_c')\,\mathrm{d}S - \sum_{N_2} \int_{S_i} \underline{n} \cdot (p'\underline{\underline{\delta}} + \underline{\underline{\tau}}_c')\,\mathrm{d}S \tag{24}$$

$$r_0^3 \underline{F}_{dc} = r_0^3 \underline{F}^*_{dc} - \sum_{N_2} \int_{S_i} \underline{n} \cdot (p'\underline{\underline{\delta}} + \underline{\underline{\tau}}_c')\,\mathrm{d}S + \sum_{N_1} \int_{S_e} \underline{n} \cdot (p'\underline{\underline{\delta}} + \underline{\underline{\tau}}_c')\,\mathrm{d}S. \tag{25}$$

In these equations S_i is that part of the surface of the dispersed particles which is facing inwardly towards the control volume element, while S_e is the part facing outwardly. n is the normal to the surface of the particles directed towards the continuous phase. It is found—as it should be—that:

$$\left[\begin{array}{c}\text{external stresses}\\ \text{on both phases}\\ \text{together}\end{array}\right]_{\mathrm{I}} = \left[\begin{array}{c}\text{external stresses}\\ \text{on both phases}\\ \text{together}\end{array}\right]_{\mathrm{II}} \tag{26}$$

while also

$$\left[\begin{array}{c}\text{external stresses}\\ \text{on cont. phase}\end{array}\right]_{\mathrm{I}} + r_0^3 \underline{F}_{dc} = \left[\begin{array}{c}\text{external stresses}\\ \text{on cont. phase}\end{array}\right]_{\mathrm{II}} + r_0^3 \underline{F}^*_{dc} \tag{27}$$

and

$$\left[\begin{array}{c}\text{external stresses}\\ \text{on disp. phase}\end{array}\right]_{\mathrm{I}} - r_0^3 \underline{F}_{dc} = \left[\begin{array}{c}\text{external}\\ \text{stresses on}\\ \text{disp. phase}\end{array}\right]_{\mathrm{II}} - r_0^3 \underline{F}^*_{dc}. \tag{28}$$

Since

$$\left[\begin{array}{c}\text{external stresses}\\ \text{on cont. phase}\end{array}\right]_{\mathrm{I}} = -(\nabla p + \nabla \cdot \underline{\underline{\tau}}_c) r_0^3$$

and

$$\left[\begin{array}{c}\text{external stresses}\\ \text{on cont. phase}\end{array}\right]_{\mathrm{II}} = -(\nabla \epsilon p + \nabla \cdot \epsilon \underline{\underline{\tau}}_c) r_0^3$$

it is found that:

$$\underline{F}_{dc} - \underline{F}^*_{dc} = \nabla(1-\epsilon)p + \nabla \cdot (1-\epsilon)\underline{\underline{\tau}}_c. \tag{29}$$

This same result is derived in a more direct way in appendix II. In section 4.1 it was found that

$$\underline{F}_{dc} = (1-\epsilon)(\nabla p + \nabla \cdot \underline{\underline{\tau}}_c) + \underline{F}_1$$

and so we now have

$$\underline{F}^*_{dc} = p\nabla\epsilon + \underline{\underline{\tau}}_c \cdot \nabla\epsilon + \underline{F}_1. \qquad [30]$$

The continuous phase momentum balance now reads:

$$\epsilon\rho_c \frac{D}{Dt}(\underline{v}_c) = -\nabla \cdot \underline{\underline{R}}_c - \nabla \cdot \epsilon\underline{\underline{\tau}}_c - \nabla\epsilon p + \epsilon\rho_c \underline{g} + \underline{F}^*_{dc} \qquad [31]$$

which after substitution of [30] reads the same as [21].

To derive now the dispersed phase momentum balance we first apply [28] and [29]. Realizing that in applying method I the external stresses on the disp. phase were absent we therefore find

$$\begin{bmatrix}\text{external stress}\\ \text{on disp. phase}\end{bmatrix}_{\text{II}} = -\nabla(1-\epsilon)p - \nabla \cdot (1-\epsilon)\underline{\underline{\tau}}_c\} r_0^3 \qquad [32]$$

which equation holds as long as there is no interaction between the dispersed particles themselves.

In applying method II we already stated that occasional or permanent contact between the dispersed particles is possible and in that case the particles outside the control volume element might exert stresses on the dispersed particles inside this volume element. The stresses in the dispersed phase together form the stress tensor $\underline{\underline{\sigma}}_d$. The total action of this tensor on the dispersed phase is indicated by $-V \cdot (1-\epsilon)\underline{\underline{\sigma}}_d$.

In section 4.2 we already mentioned the tensor $\underline{\underline{R}}_d = \rho_d\{\overline{(1-\epsilon)\underline{v}''_d\underline{v}''_d}\}$ on which we argued that it must be zero when there is no interaction between the dispersed particles. Since in general this is not the case, there remains an effective $\underline{\underline{R}}^*_d$ acting on the dispersed phase. Since, however, in practice $\underline{\underline{R}}^*_d$ can not be measured and its effect is the same as that of the stress tensor $\underline{\underline{\sigma}}_d$, we define a new stress tensor $\underline{\underline{\sigma}}^*_d$: $(1-\epsilon)\sigma_d$.

$$(1-\epsilon)\underline{\underline{\sigma}}^*_d = (1-\epsilon)\underline{\underline{\sigma}}_d + \underline{\underline{R}}^*_d.$$

The momentum balance for the dispersed phase now becomes:

$$(1-\epsilon)\rho_d \frac{D}{Dt}(\underline{v}_d) = -\nabla \cdot (1-\epsilon)\underline{\underline{\sigma}}^*_d - \nabla(1-\epsilon)p - \nabla \cdot (1-\epsilon)\underline{\underline{\tau}}_c + (1-\epsilon)\rho_d \underline{g} - \underline{F}^*_{dc}. \qquad [33]$$

Except for the first term in the r.h.s. this equation becomes the same as [22] after substituting (30):

$$(1-\epsilon)\rho_d \frac{D}{Dt}(\underline{v}_d) = -(1-\epsilon)\nabla \cdot \underline{\underline{\tau}}_c - \nabla \cdot (1-\epsilon)\underline{\underline{\sigma}}^*_d - (1-\epsilon)\nabla p + (1-\epsilon)\rho_d \underline{g} - \underline{F}_1. \qquad [34]$$

5. DISCUSSION

(1) It has been shown that both methods I and II for forming the momentum balances of a two-phase dispersed system give the same final result which gives us confidence that this result

is as good as possible for a dispersed system. One of the major assumptions was the restriction of the control volume element: $l_d \ll r_0 \ll L$. It is believed (on intuitive basis) that the size r_0 is small enough if say 100 dispersed particles are contained in the control element.

(2) Rotation of particles and internal circulation were so far not discussed. In fact there was no need since the derivation of $\underline{F}_{dc}$ as given is independent whether there is circulation/rotation or not. Of course circulation/rotation do have influence but this influence manifests itself in the fluctuations of $\underline{v}$, $\underline{\underline{\tau}}_c$ and p and hence in the formation of $\underline{\underline{R}}_c$ and of $\underline{F}_1$.

(3) The term $\underline{\underline{R}}_c$ can not be eliminated by combining it with shear stress tensor $\underline{\underline{\tau}}_c$ as done, e.g. by Anderson & Jackson (1967). This must be clear from the elaboration of the interaction term $\underline{F}_{dc}$ in which $\underline{\underline{\tau}}_c$ occurs but $\underline{\underline{R}}_c$ does not.

(4) The tensor $\underline{\underline{R}}_c$—except for the effects of rotation/circulation—accounts for (a) the effects of the Reynolds stresses on the momentum transport through the continuous phase; (b) the possible sideward drift of the dispersed particles in a non-uniform velocity profile of the continuous phase; (c) the lagging behind of dispersed particles in a non-uniform velocity profile even if these particles have the same density as the continuous phase, which occurs due to averaging of shear stress and velocity over the surface of the particles (Simha 1936).

(5) As follows from

$$\underline{F}_{dc} = (1-\epsilon)(\nabla p + \nabla \cdot \underline{\underline{\tau}}_c) + \underline{F}_1$$

the total interaction force is composed of several contributions.

It will be clear that the part $(1-\epsilon)\nabla p$ is the well-known buoyancy Archimedes force which also acts if the slip velocity is zero. A similar action upon the particles is caused by the term $(1-\epsilon)\nabla \cdot \underline{\underline{\tau}}_c$, hence remains the term:

$$\underline{F}_1 = \frac{1}{\Delta x \Delta y \Delta z} \int_{\substack{\Delta x, \Delta y, \Delta z \\ \text{dispersed phase}}} (\nabla p'' + \nabla \cdot \underline{\underline{\tau}}_c'')\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Indeed it is sure that the fluctuations p'' and $\underline{\underline{\tau}}_c''$ around the particles will mainly be brought about by the flow of the continuous phase around the particles.

It is believed, therefore, that F_1 is the so-called slip force which is directly related to and at low Reynolds numbers proportional to the slip velocity $\underline{v}_s = \underline{v}_d - \underline{v}_c$.

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NOTATION

a	radius of dispersed particle
d	diameter of dispersed particle
E_{jk}	stress tensor as defined by Anderson & Jackson
$\underline{F}'_{pc}$	interaction force of single particle on continuous phase
$\underline{F}_{dc}$	interaction force of dispersed phase on continuous phase per unit volume on dispersed system
$\underline{F}_1$	see [16]
$\underline{g}$	gravity force per unit mass

$g(l)$	weighting function of Anderson & Jackson
l	coordinate of weighting function
l_d	scale of dispersion
L	scale of equipment
$\underline{n}$	mass flux
N_0, N_1, N_2	number of dispersed particles in control volume element (see figure 2)
p	isotropic pressure
r_0	size of control volume element
$\underline{r}$	location vector
$\underline{\underline{R}}_c$	$\overline{\epsilon\rho_c v''_c v''_c}$
S	surface element of dispersed particle
S_p	surface area of dispersed particle
t	time
$\underline{v}$	linear velocity
V	volume element
V_p	volume of dispersed particle
x, y, z	space coordinates
γ	interfacial tension
$\underline{\underline{\delta}}$	unit tensor
ϵ	porosity = volume fraction of continuous phase
ρ	density
$\underline{\underline{\sigma}}_d$	stress tensor indispersed phase
$\underline{\underline{\tau}}_c$	shear stress tensor in continuous phase

Subscripts

c	continuous
d	dispersed phase
i	internal
e	external

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APPENDIX I

The averaging method of Anderson & Jackson

If a' is any point property of the continuous phase then the local mean value a at the point $\underline{r}$

is defined by:

$$\epsilon(\underline{r})a(\underline{r}) = \int_{V^\infty,\ \text{cont.phase}} a'(\underline{r}_1)g(\underline{r}-\underline{r}_1)\,\mathrm{d}V_{r_1}$$

where the integration is taken only over points which are occupied by the continuous phase. Similarly for any point property b' of the dispersed phase:

$$\{1-\epsilon(\underline{r})\}b(\underline{r}) = \int_{V^\infty,\text{disp.phase}} b'(\underline{r}_1)g(\underline{r}-\underline{r}_1)\,\mathrm{d}V_{r_1} = \sum_{p^\infty}\int_{V_p} b'(\underline{r}_1)g(\underline{r}-\underline{r}_1)\,\mathrm{d}V_{r_1}$$

where the integration is taken only over points occupied by the dispersed phase, and the summation over all dispersed particles.

In working out the local mean value of $\nabla\cdot E'_{jk}$ where E'_{jk} stands for $(p'\underline{\underline{\delta}} + \underline{\underline{\tau}}'_c)$, Anderson & Jackson arrive at their equation [32] which says that:

$$\sum_{p^\infty}\int_{S_p} \underline{n}\cdot E_{jk}g(\underline{r}-\underline{r}_1)\,\mathrm{d}S_r = \int_{V^\infty,\ \text{disp.phase}} g(\underline{r}-\underline{r}_1)\nabla\cdot E_{jk}\,\mathrm{d}V_{r_1} - \nabla\cdot\int_{V^\infty,\ \text{disp.phase}} E_{jk}g(\underline{r}-\underline{r}_1)\,\mathrm{d}V_{r_1}.$$

Next they state that both E_{jk} and $\nabla\cdot E_{jk}$ vary little over distances comparable with the radius of g and that hence they may be evaluated at $\underline{r}$ rather than at $\underline{r}_1$ and taken outside the integrals.

Now this can not be correct:

(a) If $\nabla\cdot E_{jk}$ may be considered constant over the radius of g then E_{jk} itself is not constant unless $\nabla\cdot E_{jk} = 0$.

(b) In their final result Anderson & Jackson derive that the total interaction force $\underline{F}_{dc}$ between the two phases is exactly as our result:

$$\underline{F}_{dc} = (1-\epsilon)\{\nabla p + \nabla\cdot\underline{\underline{\tau}}_c\} + \underline{F}_1$$

and hence contains the Archimedes force $(1-\epsilon)\nabla p$ which the dispersed particles experience due to the effect of the pressure gradient over the size of these particles. If this force can not be neglected (with which we agree) than how can it be neglected over the radius of the function g which radius must be much larger than that of the dispersed particles.

Because of the above arguments we tried an approach in which we assumed that at any point $\underline{r}$ local mean properties like p can be linearized so that:

$$p'(\underline{r}_1) = p(\underline{r}) + \nabla_r p\cdot(\underline{r}_1 - \underline{r}) + p''.$$

In elaborating now $\int_{V^\infty\text{cont.phase}} g(\underline{r}-\underline{r}_1)\nabla_{r_1}(p')\,\mathrm{d}V_{r_1}$, however, we met unsurmountable difficulties.

APPENDIX II

A. *Proof that* $\underline{F}_{dc} - \underline{F}^*_{dc} = \nabla(1-\epsilon)p + \nabla\cdot(1-\epsilon)\underline{\underline{\tau}}_c$

According to [25]:

$$r_0^3(\underline{F}_{dc} - \underline{F}^*_{dc}) = r_0^3\underline{F}_{dc+} = \sum_{N_1}\int_{S_e} - \sum_{N_2}\int_{S_i} \qquad [a]$$

in which $\sum_N \int_S$ stands for:

$$\sum_N \int_S \underline{n} \cdot (p'\underline{\underline{\delta}} + \underline{\underline{\tau}}'_c)\,\mathrm{d}S$$

If we put $S = S_i + S_e$ we can write:

$$\sum_{N_1+N_2} \int_S = \sum_{N_1+N_2} \int_{S_i} + \sum_{N_1} \int_{S_e} + \sum_{N_2} \int_{S_e}. \qquad \text{[b]}$$

From (a) and (b)

$$\sum_{N_1+N_2} \int_S = \sum_{N_1+N_2} \int_{S_i} + \sum_{N_2} \int_{S_e} + \sum_{N_2} \int_{S_i} + r_0^3 \underline{F}_{dc+} \qquad \text{[c]}$$

or:

$$\sum_{N_1} \int_S = \sum_{N_1+N_2} \int_{S_i} + r_0^3 \underline{F}_{dc+}. \qquad \text{[d]}$$

If we consider a particle in the interface of the cubic volume element (see figure A) then for such a particle

$$\int_{S_i} \underline{n} \cdot (p'\underline{\underline{\delta}} + \underline{\underline{\tau}}'_c)\,\mathrm{d}S = \int_{S'} \underline{n} \cdot (p'\underline{\underline{\delta}} + \underline{\underline{\tau}}'_c)\,\mathrm{d}S - \int_{S_c} \underline{n} \cdot (p'\underline{\underline{\delta}} + \underline{\underline{\tau}}'_c)\,\mathrm{d}S \qquad \text{[e]}$$

where $S' = S_i + S_c$ = the interface which encloses the shaded part of the particle in figure A. This shaded part is entirely within the cube. The volume of this part is called V'_p. (It is supposed that in writing (e) p' and $\underline{\underline{\tau}}_c'$ are the imaginary continuation inside the particle of p' and $\underline{\underline{\tau}}'_c$ outside of the particle.)

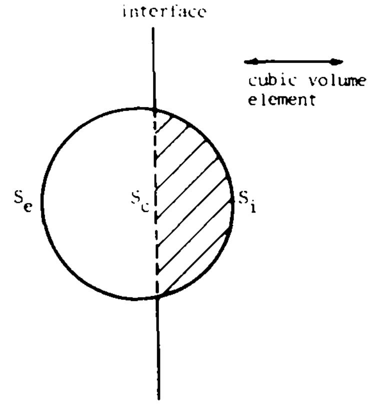

Figure A.

As both S and S' are closed surfaces the corresponding surface integrals can be transferred into volume integrals. Hence from (d) and (e) after summation over all particles $N_1 + N_2$ it follows:

$$\sum_{N_1} \int_{V_p} = \sum_{N_1+N_2} \int_{V'_p} - \sum_{N_1+N_2} \int_{S_c} + r_0^3 \underline{F}_{dc+}. \qquad \text{[f]}$$

Now it can be shown (see further on under B) that

$$N_1 V_p = \sum_{N_1+N_2} V'_p$$

hence

$$\sum_{N_1} \int_{V_p} = \sum_{N_1+N_2} \int_{V'_p} \qquad [g]†$$

and now from (f) and (g):

$$r_0^3 \underline{F}_{dc+} = \sum_{N_1+N_2} \int_{S_c} \cdot \qquad [h]$$

Evaluation of $\sum_{N_1+N_2} \int_{S_c}$ according to earlier indicated procedure gives

$$\sum_{N_1+N_2} \int_{S_c} \underline{n} \cdot (p\underline{\underline{\delta}} + \underline{\underline{\tau}}'_c)\, \mathrm{d}S = r_0^3\{\nabla(1-\epsilon)p + \nabla \cdot (1-\epsilon)\underline{\underline{\tau}}_c\} \qquad [j]$$

so that

$$\underline{F}_{dc+} = \underline{F}_{dc} - \underline{F}^*_{dc} = \nabla(1-\epsilon)p + \nabla \cdot (1-\epsilon)\underline{\underline{\tau}}_c.$$

B. *Proof that* $N_1 V_p = \sum_{N_1+N_2} V'_p$ *(see figure* 2)

It is assumed that the porosity ϵ is uniformly distributed and that all dispersed particles have the same volume V_p.

The *expectation* of the number of mass centres of dispersed particles inside a cube with edge r_0 is now $(1-\epsilon)r_0^3/V_p$.

The *actual* number is $N_0 + N_1$. If the number of particles inside the cube is large enough we may equalize these two numbers:

$$N_0 + N_1 = (1-\epsilon)r_0^3/V_p$$

or [k]

$$(N_0 + N_1)V_p = (1-\epsilon)r_0^3$$

The *expectation* of the total volume dispersed phase inside the cube is $(1-\epsilon)r_0^3$, while the actual volume dispersed phase inside the cube is

$$N_0 V_P + \sum_{N_1+N_2} V'_p.$$

Equalizing gives

$$N_0 V_p + \sum_{N_1+N_2} V'_p = (1-\epsilon)r_0^3 \qquad [l]$$

[k] and [l] together give:

$$N_1 V_p = \sum_{N_1+N_2} V'_p. \qquad [m]$$

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†This is not quite exact since in the left-hand side integration is across the interface, while in the right-hand side integration is only over that part of the particles which is inside the cube. However, it may be assumed that p and all components of $\underline{\underline{\tau}}_c$ change linearly with distance and then (g) is correct.